PHYSICAL REVIEW E **74**, 040101(R) (2006)

Contact process in heterogeneous and weakly disordered systems

C. J. Neugebauer* and S. V. Fallert
Department of Chemistry, University of Cambridge, Cambridge, United Kingdom

S. N. Taraskin
St. Catharine's College and Department of Chemistry, University of Cambridge, Cambridge, United Kingdom

(Received 29 May 2006; published 9 October 2006)

The critical behavior of the contact process (CP) in heterogeneous periodic and weakly disordered environments is investigated using the supercritical series expansion and Monte Carlo (MC) simulations. Phase-separation lines and critical exponents β (from series expansion) and η (from MC simulations) are calculated. A general analytical expression for the locus of critical points is suggested for the weak-disorder limit and confirmed by the series expansion analysis and the MC simulations. Our results for the critical exponents show that the CP in heterogeneous environments remains in the directed percolation universality class, while for environments with quenched disorder, the data are compatible with the scenario of continuously changing critical exponents.

DOI: 10.1103/PhysRevE.74.040101 PACS number(s): 05.70.Ln, 64.60.Ht, 02.50.Ey, 87.18.Bb

Phase transitions in nonequilibrium statistical mechanics have long been a field of interest, and universality classes similar to those in equilibrium have been identified, with the directed percolation (DP) class being one of the most prominent ones. The contact process (CP) [1], a susceptible-infected-susceptible model for the spread of epidemics, belongs to this universality class and has become one of its archetypical models. Recent years have seen much activity sparked by the question of whether or not the DP class is robust with respect to the introduction of quenched spatial disorder. This has been seen as an important question from both a fundamental and an experimental viewpoint, as it has been suggested that the lack of experimental observations of the DP critical behavior might be a result of the presence of disorder in real-world systems [2].

One of the foremost arguments that disorder changes the critical behavior of the CP is that it violates the Harris criterion [3,4] for all dimensions $d<4$. This criterion states that a critical point is stable with respect to disorder if $d\nu_{\perp}>2$ where $\nu_{\perp}$ is the critical exponent associated with the spatial correlation length. So far, all studies carried out on the disordered CP have provided supporting evidence for a change in universality with the introduction of disorder [5–12]. However, it is not entirely clear how the critical exponents change with disorder. In the strong-disorder limit, Hooyberghs *et al.* [10,11] have demonstrated that the CP changes to the universality class of the random transverse-field Ising model with activated scaling characterized by known scaling exponents. Recent Monte Carlo (MC) simulations [12] suggest that the activated scaling holds for an arbitrary degree of disorder, meaning that an introduction of even weak disorder forces the abrupt change of critical exponents from known values for the homogeneous CP to those of the infinite-randomness fixed point (IRFP). This contradicts the findings of other authors [8,9,11], who showed, using both MC simulations and density-matrix renormalization-group (RG) analysis, that there is an intermediate-disorder regime with continuously varying exponents. Unconventional critical behavior produced by quenched randomness is supported by a field-theoretical analysis [7] in which only runaway solutions in the RG equations were found.

The subject of this paper is to investigate the one-dimensional (1D) CP in heterogeneous periodic systems (e.g., a regular binary chain) and in systems with weak disorder (e.g., a binary chain with randomly placed species characterized by parameters close in value). In order to achieve this goal, we employ the supercritical series expansions [13,14] and MC simulations. We also suggest a simple analytical expression for the locus of critical points in the rate-space phase diagram which is in very good agreement with series expansion and MC simulation results for both heterogeneous periodic and weakly disordered systems. Our main findings demonstrate that the CP in heterogeneous periodic systems belongs to the DP universality class with the scaling exponents coinciding with those for the homogeneous case. For weakly disordered systems, we can state that the introduction of disorder does not force the exponents to change to the values of the IRFP but rather causes their continuous change with disorder strength.

The CP is usually defined on a hypercubic lattice of nodes which can be either empty (susceptible) or occupied (infected). The infection occurs via contacts between Z nearest nodes i and j with the rate λ_{ij}/Z. An infected node i can recover to susceptible one with the rate μ_i. The time scale is defined by setting all $\lambda_{ij}=1$ (for simplicity, there is no disorder in transmission rates) and, for concreteness, we consider only binary systems with two types of nodes A and B, characterized by the recovery rates μ_A and μ_B, respectively, which are distributed according to a bimodal distribution in the disordered system, $\rho(\mu_i)=(1-q)\delta(\mu_i-\mu_A)+q\delta(\mu_i-\mu_B)$, with q being the concentration of nodes B.

In the homogeneous case ($q=0$), the CP undergoes a nonequilibrium phase transition between active and absorbing states [15] if the recovery rates become greater than a critical value, $\mu>\mu_c\simeq 0.303\,228$ [14]. At criticality, the number of

*Electronic address: cjn24@cam.ac.uk

infected sites, $N_{\text{inf}}(t)$, scales with time as $N_{\text{inf}}(t) \propto t^{\eta}$, with η=0.313 686 [16]. Close to criticality, the survival probability P_∞ and the time and space correlation lengths $\xi_\parallel$ and $\xi_\perp$, respectively, also exhibit typical critical behavior, $P_\infty \propto \Delta^{\beta}$, $\xi_\parallel \propto |\Delta|^{-\nu_\parallel}$, and $\xi_\perp \propto |\Delta|^{-\nu_\perp}$ where $\Delta = \mu_c - \mu$ and the universal exponents are $\beta \simeq 0.2769$ [14], $\nu_\parallel = 1.733\,825(25)$, and $\nu_\perp = 1.096\,844(14)$ [16].

In the heterogeneous and disordered systems, a similar transition occurs [17]. Below, we address two questions: (i) how heterogeneity and disorder influence the universal properties (namely, scaling exponents) at criticality and (ii) what the locus of critical points is in the rate space of the CP in such systems. We start with simple arguments about a possible way of evaluating an analytical expression for the critical line separating the active and absorbing states.

Let us consider a system of N ($N \to \infty$) nodes characterized by random recovery rates $\tilde{\mu}_i = \mu_i/\mu_c$. Following the formalism developed in Refs. [18–20], the state of the system is described by the state vector $|P(t)\rangle = \Sigma_{\{\sigma\}} P(\{\sigma\},t)|\{\sigma\}\rangle$, whose time evolution is governed by the master equation $\partial_t |P(t)\rangle = \hat{L}|P(t)\rangle$. Here $\hat{L}$ is the generator of the Markov process that contains the transition rates between the different microstates of the system, $|\{\sigma\}\rangle$. Once the system approaches the active state, one of the nontrivial eigenvalues approaches zero (and becomes exactly zero at the critical point in the infinite system). This means that the free term $F(\tilde{\mu}_i)$ in the characteristic equation $\lambda^{-1}|\lambda \hat{I} - \hat{L}| = 0$ also approaches zero at criticality. Therefore, the locus of critical points in the space of recovery rates can be written in a general form as a solution of the equation $F(M_i) \simeq 0$ (for $i = 1, 2, \ldots, N$), where the choice of the argument $M_i = \ln \tilde{\mu}_i$ is motivated by a similar choice for the RG analysis of both the CP and random transverse-field Ising model [10,21]. The function $F(M_i)$ is invariant under the exchange of any two arguments and it obeys the property $F(0) \simeq 0$ for the homogeneous case. As this function is analytic around the homogeneous critical point at any finite N we can expand it in a Taylor series around this point, $F'(0)\Sigma_i^N M_i + O(M_i^2) \simeq 0$, where we have used the symmetry of the function F leading to all first derivatives being equal at the stationary point. In fact, this expansion is equivalent to the expansion in the moments $u_n = E[M^n]$, where $E[\cdot]$ denotes the expectation value. Leaving only the first order in the above expansion we end up with the following approximate equation for the locus of critical points:

$$E[\ln \tilde{\mu}] \simeq 0, \tag{1}$$

which is valid around the homogeneous critical point.

In order to support Eq. (1) for the locus of critical points and also to investigate the critical behavior of the model, we have used the perturbative supercritical series expansion [13,14] for the survival probability P_∞. As usual, $\hat{L} = \mu\hat{W} + \hat{V}$, is split into a part that destroys particles, $\mu\hat{W}$, and a part that creates particles, $\hat{V}$, which in the systems under consideration take the forms

$$\mu\hat{W} = \sum_i \mu_i (1 - a_i^\dagger) a_i, \tag{2}$$

$$\hat{V} = \sum_i \frac{1}{2}(1 - a_i) a_i^\dagger (a_{i-1}^\dagger a_{i-1} + a_{i+1}^\dagger a_{i+1}), \tag{3}$$

where $a_i^\dagger$ and a_i are hard-core bosonic creation and annihilation operators, respectively. The Laplace transform of $|P(t)\rangle$, $|\tilde{P}(s)\rangle = (s - \mu\hat{W} - \hat{V})^{-1}|P(0)\rangle$, is then expanded in μ_i, yielding the supercritical expansion for the survival probability $P_\infty(\mu_A, \mu_B) = \lim_{s\to 0}[1 - s\langle 0|\tilde{P}(s)\rangle]$, where $|0\rangle$ is the absorbing state. For the analysis of this multivariable survival probability (cf. Ref. [22]), we employ a numerical scheme similar to the nested Padé approximation [23,24]. In order to investigate the critical behavior, we consider the meromorphic function $\partial_{\mu_A} \ln P_\infty(\mu_A, \delta)$ (with $\delta = \mu_B - \mu_A$), whose first pole on the positive real axis is the critical point of the model, and the residue at that pole is the critical exponent β. To improve estimates of these poles from the finite series expansion, the following multivariable rational-approximant scheme is used: for a given expansion of P_∞ in two variables μ_A and δ up to an even (odd) order N, the Padé approximants $[n,n]$ ($[n,n+1]$) in δ of the coefficients of the terms μ_A^{N-1-2n} (μ_A^{N-2n}) in the series $\partial_{\mu_A} \ln P_\infty(\mu_A, \delta)$ are formed, followed by the construction of the Padé approximant $[N/2-1, N/2]$ ($[(N-1)/2, (N-1)/2]$) in μ_A. In order to estimate the stability of the poles and residues found, several Padé approximants in μ_A (e.g., the approximants from $[N/2-1, N/2]$ down to $[N/2-2, N/2-1]$ for even orders of N) were constructed and averaged over.

Using this scheme and starting from a single seed in the series expansions up to order N=24, we calculated the locus of critical points and the critical exponents β for three heterogeneous lattices AB, AAB, and $AABB$, and for disordered systems whose recovery rates are drawn from the bimodal distribution mentioned above (see Figs. 1 and 2). Figure 1(a) demonstrates that around the homogeneous critical point (μ_c, μ_c), the phase-separation lines between the active and absorbing states are indeed very well described by Eq. (1). This is also confirmed by single-seed MC simulations based on the random-sequential algorithm [25] (see Table I)—the deviations of the MC results from predictions of Eq. (1) and series expansion data for the critical line are less than 1%. The critical values of μ_B for fixed values of μ_A were found by analysis of the power-law behavior of $N_{\text{inf}}(t)$ with averaging over 10^6 MC runs. Figure 1(b) and Table I also show that the critical exponents β and η practically do not change from the values for the homogeneous CP, thus confirming that the CP in heterogeneous lattices belongs to the same universality class, DP, as that in the homogeneous one. The systematic deviations of the calculated critical rates from the theoretical prediction increases with the distance from the homogeneous point, thus reflecting the restricted range of applicability of Eq. (1). Some irregular fluctuations in both μ_B and β are probably due to poor convergence of the series expansions.

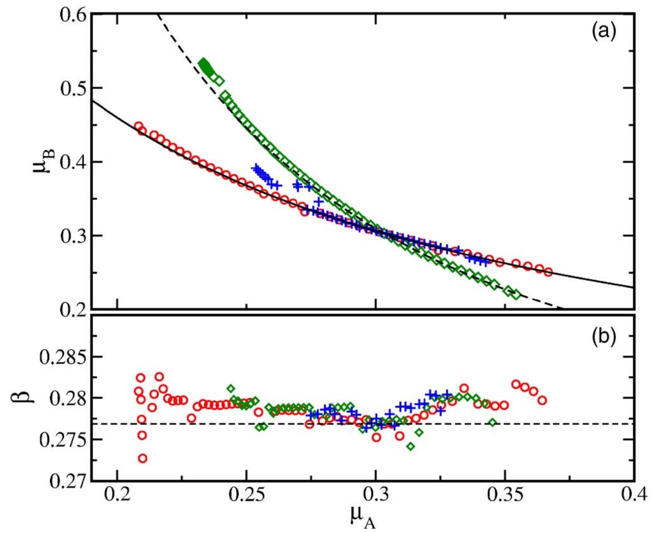

FIG. 1. (Color online) Periodic 1D lattices *AB* (○), *AAB* (◇), and *AABB* (+): (a) critical points obtained by series expansions in comparison with analytical prediction for the critical line $\mu_c=(\mu_A^{1-q}\mu_B^q)$, $q=1/2$ (—) and $q=1/3$ (- - -) for $\mu_c=0.303\,228$ [14]; (b) critical exponent β in comparison with series expansion value $\beta=0.2769$ (- - -) [14] for the homogeneous case.

A similar series expansion analysis has been performed for disordered systems with a configurational averaging of the survival probability $\langle P_\infty\rangle$. We were able to perform the complete numerical averaging over all 2^{2N-1} configurations for $N\leqslant 12$. The configurational averaging for series expansions to higher orders, $N=19$, has been done approximately.

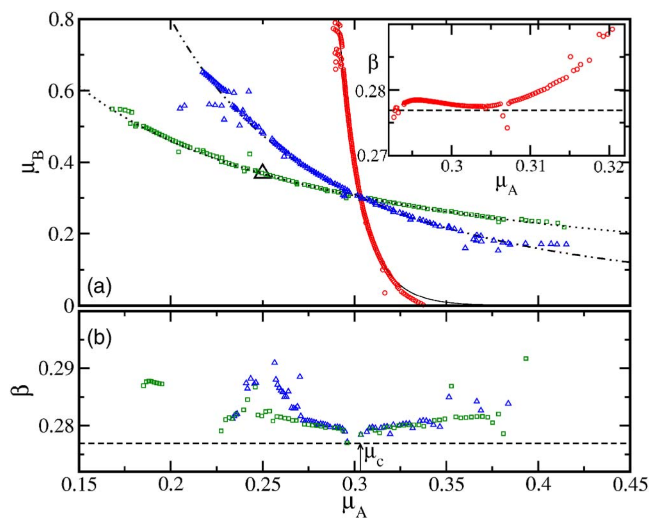

FIG. 2. (Color online) The phase diagram (a) and scaling exponent $\beta(\mu_A)$, (b) and inset of (a), for disordered lattices with various degrees of disorder: $q=0.04$ (○), 0.3 (△), and 0.5 (□) obtained by series expansion. The lines represent the theoretical prediction according to Eq. (1), $\mu_c=(\mu_A^{1-q}\mu_B^q)$, for $q=0.04$ (—), 0.3 (·· - ··), and 0.5 (···) with $\mu_c=0.303\,228$ [14]. The triangle in (a) marks the region for which the MC simulations shown in Fig. 3 have been run. The dashed lines in (b) and the inset of (a) show the value of β for the homogeneous case, $\beta_c=0.2769$ (- - -) [14]. The arrow in (b) marks the homogeneous critical point.

TABLE I. The critical values of μ_B obtained in MC simulations (second column) and calculated according to Eq. (1) (third column) together with the MC critical exponents (fourth column) for fixed values of μ_A (first column) in the heterogeneous systems *AB* and *AAB*.

	μ_A	μ_B (MC)	μ_B (predicted)	η
AB	0.2750	0.3344±0.0001	0.3343	0.313±0.006
	0.2500	0.3681±0.0001	0.3678	0.313±0.003
	0.2250	0.4094±0.0001	0.4087	0.313±0.003
AAB	0.2750	0.3689±0.0001	0.3687	0.313±0.002
	0.2500	0.4475±0.0001	0.4461	0.314±0.001
	0.2250	0.5546±0.0001	0.5507	0.314±0.001

In the calculation of $\langle P_\infty\rangle=\Sigma_{n=0}^N\Sigma_{m=0}^n\langle c_{nm}\rangle\mu_B^m\mu_A^{n-m}$, at each term of order $M\leqslant N$, we only included realizations with number $i\leqslant i_M$ of "impurity" sites B; e.g., we have chosen $i_{N-n}=n+2$ for series expansions up to order $N=19$, so that for $M=18$ only realizations with up to three impurities contributed to $\langle P_\infty\rangle$. Dropping these disorder realizations from the configurational average incurs less error the smaller the impurity concentration q is. Assuming that the coefficients $c_{Mm}\sim c_M$ (with c_M being the coefficient of the same order in the homogeneous case), all the realizations with i impurities contribute a term $\binom{2M-1}{i}(1-q)^{2M-1-i}q^i$ to the configurational average, $\langle c_{Mm}\rangle$. Then, it can easily be shown that for $q=q_{\max}=0.04$ all terms with $i>i_N=2$ are smaller than the terms with $i\leqslant i_N$. The validity of this approximation has been confirmed by testing it against the exact results for $N\leqslant 12$.

The results for fully and partially averaged survival probabilities in disordered systems are shown in Fig. 2. The phase-separation lines have been obtained for arbitrary impurity concentration for the fully averaged P_∞ expanded up to order $N=12$ and two of them for $q=0.5$ (squares and dotted line) and $q=0.3$ (triangles and dot-dashed line) are displayed in Fig. 2(a). High-order series expansions ($N=19$) have been calculated only for low impurity concentrations, $q\leqslant q_{\max}=0.04$ [see the circles and solid line in Fig. 2(a)]. Again, the poles of $\partial_{\mu_A}\ln P_\infty(\mu_A,\delta)$ agree very well with the theoretical prediction given by Eq. (1).

The residues of the poles (exponents β) for different points on the critical line are shown in Fig. 2(b) and in the inset in Fig. 2(a). The value of β reaches a minimum $\beta_{\min}$ located approximately around the homogeneous critical point (μ_c,μ_c), with $\beta_{\min}$ being rather close to the value of the homogeneous critical exponent β_c, with $|\beta_{\min}-\beta_c|/\beta_c\simeq 0.5\%$ for $N=12$ and 0.21% for $N=19$ [see the inset in Fig. 2(a)], thus confirming that the value of the exponent is much more sensitive to N than the critical rates. Away from the critical point, the value of β, first monotonically increases and then starts to fluctuate due to a high sensitivity to the value of μ_B, the estimates of which lose precision due to poor convergence of the series in this range. The results for β are in reasonable agreement with findings in Refs. [6,9,11] where continuously varying critical exponents were seen in MC simulations and density-matrix renormalization-group studies of the random CP. Unfortunately, the errors in the

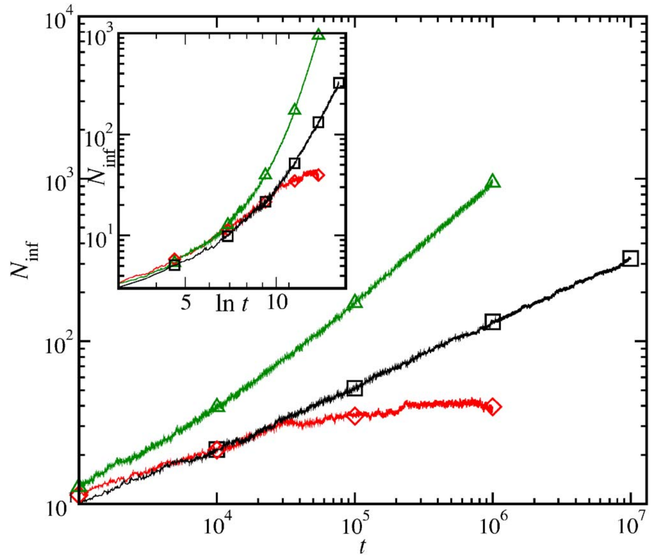

FIG. 3. (Color online) Number of infected sites vs time obtained by MC simulations of the disordered CP with $q=0.5$ and $\mu_A=0.25$, averaged over 1500 runs and at least 30 disorder realizations. The main figure is in double-logarithmic scale according to the conventional scaling while the inset demonstrates the same data in the activated scaling picture [12]. The curves from top to bottom correspond to 0.3628 (△), 0.3680 (□), and 0.3728 (◇).

exponents that are shown in Fig. 2(b) and the inset of Fig. 2(a) are at least of the order of $|\beta(\mu_c)-\beta_c|$ and the monotonic growth of the exponents found in the series expansions can be questioned. However, our results are certainly not consistent with the scenario presented in [12] according to which the weakly disordered CP belongs to the same universality class as random transverse-field Ising model with $\beta=0.381\,97$.

The results for the disorder case have been supported by MC simulations (see Fig. 3). Due to the long relaxation times of the disordered CP (cf. Refs. [5,9,12]) we have focused only on one point in the rate space with $q=0.5$ and $\mu_A=0.25$. The results of the simulations up to 10^7 time steps are shown in Fig. 3 for three values of μ_B around criticality in double-logarithmic scales of $N_{\rm inf}$ vs t as well as $N_{\rm inf}$ vs $\ln t$ (see the inset in Fig. 3) to allow for both conventional and activated scaling [12]. The MC critical value of $\mu_B\simeq 0.368$ (with the error being less than 0.005) obtained by conventional double-logarithmic scaling analysis is certainly very close to the series expansion value ($\mu_B\simeq 0.369$ for $N=12$; see the triangle in Fig. 2). The value of the dynamical exponent at this point is found to be $\eta\simeq 0.388$ which is in favour of the scenario suggesting scaling exponents varying continuously with disorder.

In conclusion, we have investigated the CP in heterogeneous and disordered 1D systems in the limit of weak disorder by means of the series expansions and MC simulations. We have demonstrated that the CP in heterogeneous 1D lattices stays in the DP universality class. For disordered environment, our results suggest that disorder continuously changes the scaling exponents. A simple analytical formula for the phase-separation line has been suggested and proved (numerically) to be valid in the weak-disorder limit. Preliminary investigations of the CP in 2D heterogeneous lattices also support the analytical predictions for the phase-separation line.

The authors would like to thank I. Jensen, B. Simons, J. Stilck, and R. Dickman for their helpful correspondence and remarks. The computations were mainly performed on the Cambridge University Condor Grid and the Cambridge-Cranfield High-Performance Computer facility. C.J.N. and S.V.F. would like to thank the EPSRC and the Cambridge European Trust for financial support.

[1] T. Harris, Ann. Probab. **2**, 969 (1974).
[2] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[3] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[4] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).
[5] A. J. Noest, Phys. Rev. Lett. **57**, 90 (1986).
[6] A. G. Moreira and R. Dickman, Phys. Rev. E **54**, R3090 (1996).
[7] H. K. Janssen, Phys. Rev. E **55**, 6253 (1997).
[8] R. Cafiero, A. Gabrielli, and M. A. Muñoz, Phys. Rev. E **57**, 5060 (1998).
[9] R. Dickman and A. G. Moreira, Phys. Rev. E **57**, 1263 (1998).
[10] J. Hooyberghs, F. Iglói, and C. Vanderzande, Phys. Rev. Lett. **90**, 100601 (2003).
[11] J. Hooyberghs, F. Iglói, and C. Vanderzande, Phys. Rev. E **69**, 066140 (2004).
[12] T. Vojta and M. Dickison, Phys. Rev. E **72**, 036126 (2005).
[13] R. Dickman and I. Jensen, Phys. Rev. Lett. **67**, 2391 (1991).
[14] I. Jensen and R. Dickman, J. Stat. Phys. **71**, 89 (1993).
[15] T. Liggett, *Interacting Particle Systems*, 1st ed. (Springer, Berlin, 1985).
[16] I. Jensen, J. Phys. A **29**, 7013 (1996).
[17] M. Bramson, R. Durrett, and R. Schonmann, Ann. Probab. **19**, 960 (1991).
[18] M. Doi, J. Phys. A **9**, 1479 (1976).
[19] P. Grassberger and A. De La Torre, Ann. Phys. (N.Y.) **122**, 373 (1979).
[20] L. Peliti, J. Phys. (France) **46**, 1469 (1985).
[21] D. S. Fisher, Phys. Rev. B **51**, 6411 (1995).
[22] W. G. Dantas and J. F. Stilck, J. Phys. A **38**, 5841 (2005).
[23] P. Guillaume, J. Comput. Appl. Math. **82**, 149 (1997).
[24] P. Guillaume and A. Huard, J. Comput. Appl. Math. **121**, 197 (2000).
[25] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models*, 1st ed. (Cambridge University Press, Cambridge, U.K., 1999).